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A GENERALIZATION OF COLEMAN'S ISOMORPHISM

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1. **General Notation.** Fix a compatible system $(1, \varepsilon_1, \ldots, \varepsilon_n, \ldots)$ of roots of unity, with $\varepsilon_{n+1}^p = \varepsilon_n$ and $\varepsilon_1 \neq 1$. If K is a finite extension of $\mathbf{Q}_p$ and $n \in \mathbf{N}$, let $K_n = K(\varepsilon_n)$ and $K_\infty = \cup_{n\in\mathbf{N}} K_n$. Let also $\mathscr{G}_K$ be the Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$ and $\chi : \mathscr{G}_K :\to \mathbf{Z}_p^*$ be the cyclotomic character and denote by $\mathscr{H}_K \subset \mathscr{G}_K$ its kernel. Finally, let $\Gamma_K = \mathscr{G}_K/\mathscr{H}_K = \mathrm{Gal}(K_\infty/K)$ and $\Lambda_K = \mathbf{Z}_p[[\Gamma_K]]$ be the completed group algebra of Γ_K.

2. **Coleman's isomorphism.** If $K = \mathbf{Q}_p$ and $u = (u_n)_{n\in\mathbf{N}}$ is an element of the projective limit of the groups $\mathscr{O}_{K_n}^*$ with respect to the norm maps, Coleman proved [5] that there exists a unique element $\mathrm{Col}_u(T)$ of $(\mathbf{Z}_p[[T]])^*$ such that $\mathrm{Col}_u(\varepsilon_n - 1) = u_n$ for all $n \in \mathbf{N}$. Now, as $\mathrm{Col}_u(T) \in (\mathbf{Z}_p[[T]])^*$, its logarithmic derivative has coefficients in $\mathbf{Z}_p$ and there is a unique measure μ_u on $\mathbf{Z}_p$ such that

$$(1) \qquad \int_{\mathbf{Z}_p} (1+T)^x \mu_u = (1+T)\frac{d}{dT}\log\big(\mathrm{Col}_u(T)\big).$$

Restricting this measure to $\mathbf{Z}_p^*$ and pulling it back to Γ_K using the cyclotomic character gives us a map from $\varprojlim \mathscr{O}_{K_n}^*$ to Λ_K which is almost an isomorphism and is known as Coleman's isomorphism. Moreover, the measure giving the Kubota-Leopoldt zeta function is the image of the cyclotomic units via this map and so Coleman's isomorphism can be thought of as a machine producing p-adic L-functions out of compatible systems of units.

All this can be thought of as being related to the p-adic representation $\mathbf{Q}_p(1)$. It seems therefore interesting to try to generalize as much as possible the results to other p-adic representations. A big breaktrough has been made by Perrin-Riou [10] in the case where the representation is crystalline and K unramified over $\mathbf{Q}_p$ using p-adic interpolation of the exponentials of Bloch-Kato [1] for the twists of the representation by powers of the cyclotomic character. Her construction has been refined by Kato-Kurihara and Tsuji in their work on trivial zeroes of p-adic L-functions and generalized to the case of de Rham representations in [6]. As explained below, the theory of (φ, Γ)-modules introduced by Fontaine [7] gives such a generalization without any restriction on the representation.

3. **The Iwasawa module attached to a p-adic representation.** Define

$$H^1_{\mathrm{Iw}}(K, V) = H^1(K, \Lambda_K \otimes V).$$

This paper is a short summary of the talk I gave at the conference and I would like to take the opportunity to thank the organizers for their invitation.

One can view $\Lambda_K \otimes V$ as the space of measures on Γ_K with values in V which makes it possible to define maps

$$\begin{aligned} H^1_{\mathrm{Iw}}(K,V) &\longrightarrow H^1(K_n, V(k)) \\ \mu &\longrightarrow \int_{\Gamma_{K_n}} \chi(x)^k \mu \end{aligned}$$

for any $n \in \mathbf{N}$ and $k \in \mathbf{Z}$. If T is a $\mathbf{Z}_p$-lattice in V which is stable under the action of $\mathscr{G}_K$, one can show, using Shapiro's lemma, that the map

$$\begin{aligned} H^1_{\mathrm{Iw}}(K,V) &\longrightarrow \mathbf{Q}_p \otimes \left(\varprojlim H^1(K_n, T(k))\right) \\ \mu &\longrightarrow \left(\ldots, \int_{\Gamma_{K_n}} \chi(x)^k \mu, \ldots\right) \end{aligned}$$

is an isomorphism for all $k \in \mathbf{Z}$ (the inverse limit above is taken with respect to corestriction maps). If $V = \mathbf{Q}_p(1)$, Kummer's theory gives us a natural map from K_n^* to $H^1(K_n, \mathbf{Z}_p(1))$ and, taking inverse limits, a map

$$\delta : \varprojlim \mathscr{O}^*_{K_n} \to H^1_{\mathrm{Iw}}(K, \mathbf{Q}_p(1)).$$

4. **(φ, Γ)-modules and Coleman's isomorphism.** The theory of (φ, Γ)-modules attaches to a p-adic representation V a module $D(V)$ with commuting actions of Γ_K and a Frobenius endomorphism φ. One of the nice features of this theory is that it is possible to reconstruct V from $D(V)$ which is a priori a simpler object. One natural problem is therefore to read directly on $D(V)$ the properties of V. One of the things that one can recover in this way is the Galois cohomology of V (cf. [8]). Using these results, it is possible to construct (cf. [3]) a natural map $\mathrm{Exp}^* : H^1_{\mathrm{Iw}}(K,V) \to D(V)$.

To relate the above construction to Coleman's, let $\mathbf{B}_{\mathbf{Q}_p}$ be the ring of Laurent series $x = \sum_{n \in \mathbf{Z}} a_n \pi^n$ where a_n is a bounded sequence of elements of $\mathbf{Q}_p$ going to 0 when n goes to $-\infty$. This ring is given an action of φ and Γ via the formulae

$$\gamma(\pi) = (1+\pi)^{\chi(\gamma)} - 1 \text{ and } \varphi(\pi) = (1+\pi)^p - 1.$$

Now, if $K = \mathbf{Q}_p$ and $V = \mathbf{Q}_p(1)$, then $D(V)$ is the $\mathbf{B}_{\mathbf{Q}_p}$-module of rank 1 with action of Γ twisted by χ and the following identity holds if $u \in \varprojlim \mathscr{O}^*_{K_n}$

$$\mathrm{Exp}^*(\delta(u)) = (1+\pi)\frac{d}{d\pi}\log(\mathrm{Col}_u(\pi)),$$

which shows that this map Exp^* is a direct generalization of Coleman's isomorphism.

5. **Relation with Bloch-Kato exponential map.** Using the theory of overconvergent representations and especially the fact that any p-adic representation of $\mathscr{G}_K$ is overconvergent [2], it is possible to relate invariants coming from the theory of (φ, Γ)-modules to invariants involving the ring $\mathbf{B}_{\mathrm{dR}}$ of p-adic periods. More precisely, the ring $\mathbf{B}_{\mathrm{dR}}$ and the ring $\mathbf{B}$ occuring in the theory of (φ, Γ)-modules are both built up from the ring of Witt vectors of the perfectization of $\mathscr{O}_{\mathbf{C}_p}/p$ and overconvergent elements in $\mathbf{B}$ are, by definition, elements x such that $\varphi^{-n}(x)$ has a meaning in $\mathbf{B}_{\mathrm{dR}}$ for n big enough.

Proposition. *If V is a de Rham representation of V and $\mu \in H^1_{\mathrm{Iw}}(K,V)$, then* $\mathrm{Exp}^*(V)$ *is overconvergent and, if n is big enough, the following identity holds in* $(\mathbf{B}^+_{\mathrm{dR}} \otimes V)^{\mathscr{H}_K}$.

$$p^{-n}\varphi^{-n}(\mathrm{Exp}^*(\mu)) = \sum_{k\in\mathbf{Z}} \exp^*\Big(\int_{\Gamma_{K_n}} \chi(x)^{-k}\Big) \tag{2}$$

Remark. (i) As mentioned above, $\int_{\Gamma_{K_n}} \chi(x)^{-k}$ is an element of $H^1(K_n, V(-k))$ and

$$\exp^* : H^1(K_n, V(-k)) \to D_{\mathrm{dR}}(V(-k)) = t^k D_{\mathrm{dR}}(V)$$

is the map constructed by Kato [9] and is dual to the exponential of Bloch and Kato [1] for the representation $V^*(1+k)$.

(ii) The term $\mathrm{CW}_{k,n}(\mu)$ corresponding to $\exp^*\left(\int_{\Gamma_{K_n}} \chi(x)^{-k}\right)$ in the sum above can be defined directly from $\mathrm{Exp}^*(\mu)$ without any reference to $\exp^*$ and the maps $\mu \to \mathrm{CW}_{k,n}(\mu)$ are generalizations of the Coates-Wiles homomorphisms [4]. Thus, formula (2) shows that they are related to Bloch-Kato's exponential maps. This last fact is usually thought of as an explicit reciprocity law.

References

[1] S. Bloch and K. Kato, L functions and Tamagawa numbers of motives, in "*The Grothendieck Fesschrift*", vol. 1, 333-400, Prog. Math., vol. **86**, Birkhaüser 1990

[2] F. Cherbonnier and P. Colmez , Représentations p-adiques surconvergentes, to appear in Inv. Math.

[3] F. Cherbonnier and P. Colmez, Théorie d'Iwasawa des représentations p-adiques d'un corps local, prépublication du LMENS-97-27, 1997

[4] J. Coates and A. Wiles, On p-adic L-functions and elliptic units, J. Australian Math. Soc., **A 26**, 1-25, 1978

[5] R. Coleman, Division values in local fields, Inv. Math. **53**, 91-116, 1979

[6] P. Colmez, Théorie d'Iwasawa des représentations de de Rham d'un corps local, to appear in Ann. of Maths

[7] J.-M. Fontaine, Représentations p-adiques des corps locaux, in "*The Grothendieck Festschrift*", vol II, Birkhauser, Boston

[8] L. Herr, *Cohomologie Galoisienne des corps p-adiques*, thèse de l'université d'Orsay, 1995

[9] K. Kato, Lectures on the approach to Iwasawa theory for Hasse-Weil L-functions via $\mathbf{B}_{\mathrm{dR}}$, in "*Arithmetic Algebraic Geometry*", Lecture Notes in math. **1553**, 1993

[10] B. Perrin-Riou, Théorie d'Iwasawa des représentations p-adiques sur un corps local. Inv. Math. **115**, 81-149, 1994

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